ECE 604, Lecture 20

Mon, Feb 25, 2019

Contents

Printed on March 24, 2019 at 16 : 47: W.C. Chew and D. Jiao.

1 Circular Waveguides, Contd.

1.1 An Application of Circular Waveguide

When a real-world waveguide is made, the wall of the metal waveguide is not made of perfect electric conductor, but with some metal with finite conductivity. Hence, tangential **E** field is not zero on the wall, and energy can dissipate into the waveguide wall. It turns out that due to symmetry, the TE_{01} of a circular waveguide has the lowest loss of all the waveguide modes including rectangular waveguide modes. Hence, this waveguide mode is of interest to astronomers who are interested in building low-loss and low-noise systems. The TE_{01} mode has electric field given by $\mathbf{E} = \hat{\phi} E_\phi$. Furthermore, looking at the magnetic field, the current is mainly circumferential flowing in the ϕ direction.

Figure 3 shows two ways of engineering a circular waveguide so that the TE_{01} mode is enhanced: by using a mode filter that discourages the guidance of other modes but not the TE_{01} mode, and second, by designing ridged waveguide wall to discourage the flow of axial current and hence, the propagation of the non-TE_{01} mode.

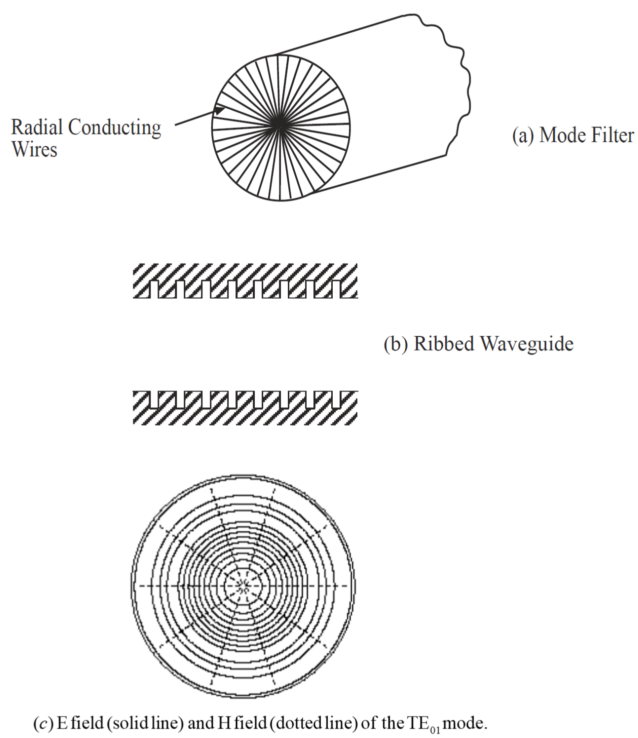

Figure 1:

2 Remarks on Quasi-TEM Modes, Hybrid Modes, and Surface Plasmonic Modes

We have analyzed some simple structures where closed form solutions are available. These solutions offer physical insight into how waves are guided, and how they are cutoff from guidance. For some simple waveguides, the modes can be divided into TEM, TE, and TM modes. However, most waveguides are not simple. We will remark on various complexities that arise in real world applications.

2.1 Quasi-TEM Modes

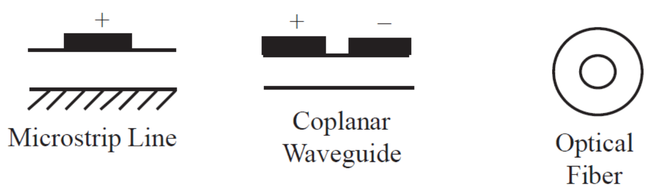

Figure 2:

Many waveguides cannot support a pure TEM mode even when two conductors are present. For example, two pieces of metal make a transmission line, and in the case of a circular coax, a TEM mode can propagate in the waveguide. However, most two-metal transmission lines do not support a pure TEM mode but a quasi-TEM mode. When a wave is TEM, it is necessary that the wave propagates with the phase velocity of the medium. But when a uniform waveguide has inhomogeneity in between, as shown in Figure 2, this is not possible anymore, and only a quasi-TEM mode can propagate. We can prove this assertion by *reductio ad absurdum* as before. From eq. (1.16) of Lect. 18, we have shown that for a TM mode, E_z is given by

$$E_z \sim (\beta^2 - \beta_z^2)\Psi_e \tag{2.1}$$

If this mode becomes TEM, then $E_z = 0$ and this is possible only if $\beta_z = \beta$. In other words, the phase velocity of the waveguide mode is the same as a plane TEM wave in the same medium.

Now assume that a TEM wave exists in both regions of the microstrip line or all three dielectric regions of the optical fiber in Figure 2, then the phase velocities in the z direction, determined by ω/β_z of each region will be ω/β_i of the respective region where β_i is the wavenumber of the i-th region. Hence, phase matching is not possible, and the boundary condition cannot be satisfied at the dielectric interfaces. Nevertheless, the lumped element model of the transmission line is still a very good model for such a waveguide. If the line capacitance and line inductances of such lines can be estimated, β_z can still be estimated.

2.2 Hybrid Modes

For most inhomogeneously filled waveguides, the modes inside are not cleanly classed into TE and TM modes, but with some modes that are the hybrid of TE and TM modes. In this case, both TE and TM waves are coupled together and are present simultaneously. In other words, both $E_z \neq 0$ and $H_z \neq 0$. Sometimes, the hybrid modes are called EH or HE modes, as in an optical fiber. Nevertheless, the guidance is via a bouncing wave picture, where the bouncing waves are reflected off the boundaries of the waveguides. In the case of an optical fiber or a dielectric waveguide, the reflection is due to total internal reflection. But in the case of metalic waveguides, the reflection is due to the metal walls.

2.3 Guidance of Modes

Propagation of a plane wave in free space is by the exchange of electric stored energy and magnetic stored energy. So the same thing happens in a waveguide. For example. in the transmission line, the guidance is by the exchange of electric and magnetic stored energy via the capacitance and the inductance of the line. In this case, the waveguide size, like the cross-section of a coaxial cable, can be made much smaller than the wavelength.

In the case of hollow waveguides, the exchange of energy stored is via the space that stores these energies, like that of a plane wave. These waveguides work only when these plane waves can enter the waveguide. Hence, the size of these waveguides has to be about half a wavelength.

The surface plasmonic waveguide is an exception in that the exchange is between the electric field stored energy with the kinetic energy stored in the moving electrons in the plasma instead of magnetic energy stored. Therefore, the dimension of the waveguide can be very small compared to wavelength, and yet the surface plasmonic mode can be guided.

3 Homomorphism of Waveguides and Transmission Lines

Previously, we have demonstrated mathematical homomorphism between plane waves in layered medium and transmission lines. Such homomorphism can be further extended to waveguides and transmission lines. We can show this first for TE modes in a hallow waveguide, and the case for TM modes can be established by invoking duality principle.

3.1 TE Case

For this case, $E_z = 0$, and from Maxwell's equations

$$\nabla \times \mathbf{H} = j\omega\varepsilon \mathbf{E} \tag{3.1}$$

By letting $\nabla = \nabla_s + \nabla_z$, $\mathbf{H} = \mathbf{H}_s + \mathbf{H}_z$ where $\nabla_z = \hat{z}\frac{\partial}{\partial z}$, and $\mathbf{H}_z = \hat{z}H_z$, and the subscript s implies transverse to z components, then

$$(\nabla_s + \nabla_z) \times (\mathbf{H}_s + \mathbf{H}_z) = \nabla_s \times \mathbf{H}_s + \nabla_z \times \mathbf{H}_s + \nabla_s \times \mathbf{H}_z \tag{3.2}$$

where it is understood that $\nabla_z \times \mathbf{H}_z = 0$. Notice that the first term on the right-hand side of the above is pointing in the z direction. Therefore, by letting $\mathbf{E} = \mathbf{E}_s + \mathbf{E}_z$, and equating transverse components in (3.1), we have

$$\nabla_z \times \mathbf{H}_s + \nabla_s \times \mathbf{H}_z = j\omega\varepsilon \mathbf{E}_s \tag{3.3}$$

Next, from Faraday's law, we have

$$\nabla \times \mathbf{E} = -j\omega\mu \mathbf{H} \tag{3.4}$$

Again, by letting $\mathbf{E} = \mathbf{E}_s + \mathbf{E}_z$, we can show that (3.4) can be written as

$$\nabla_s \times \mathbf{E}_s + \nabla_z \times \mathbf{E}_s + \nabla_s \times \mathbf{E}_z = -j\omega\mu(\mathbf{H}_s + \mathbf{H}_z) \tag{3.5}$$

Equating z components of the above, we have

$$\nabla_s \times \mathbf{E}_s = -j\omega\mu \mathbf{H}_z \tag{3.6}$$

Using (3.6), Eq.(3.3) can be rewritten as

$$\nabla_z \times \mathbf{H}_s + \nabla_s \times \frac{1}{-j\omega\mu} \nabla_s \times \mathbf{E}_s = +j\omega\varepsilon \mathbf{E}_s \tag{3.7}$$

But

$$\nabla_s \times \nabla_s \times \mathbf{E}_s = \nabla_s(\nabla_s \cdot \mathbf{E}_s) - \nabla_s \cdot \nabla_s \mathbf{E}_s \tag{3.8}$$

and since $\nabla \cdot \mathbf{E} = 0$, and $E_z = 0$ for TE modes, it implies that $\nabla_s \cdot \mathbf{E}_s = 0$. Also, from Maxwell's equations, we have previously shown that for a homogeneous source-free medium,

$$(\nabla^2 + \beta^2)\mathbf{E} = 0 \tag{3.9}$$

or that

$$(\nabla^2 + \beta^2)\mathbf{E}_s = 0 \tag{3.10}$$

Assuming that we have a guided mode, then

$$\mathbf{E}_s \sim e^{\mp j\beta_z z}, \qquad \frac{\partial^2}{\partial z^2}\mathbf{E}_s = -{\beta_z}^2 \mathbf{E}_s \tag{3.11}$$

Therefore, (3.10) becomes

$$(\nabla_s{}^2 + \beta^2 - \beta_z{}^2)\mathbf{E}_s = 0 \tag{3.12}$$

or that

$$(\nabla_s{}^2 + \beta_s{}^2)\mathbf{E}_s = 0 \tag{3.13}$$

where $\beta_s^2 = \beta^2 - \beta_z$ is the transverse wave number. Consequently, from (3.8)

$$\nabla_s \times \nabla_s \times \mathbf{E}_s = -\nabla_s^2 \mathbf{E}_s = \beta_s^2 \mathbf{E}_s \tag{3.14}$$

As such, (3.7) becomes

$$\begin{aligned}\nabla_z \times \mathbf{H}_s &= j\omega\varepsilon\mathbf{E}_s + \frac{1}{j\omega\mu}\nabla_s \times \nabla_s \times \mathbf{E}_s \\ &= j\omega\varepsilon\mathbf{E}_s + \frac{1}{j\omega\mu}\beta_s{}^2\mathbf{E}_s \\ &= j\omega\varepsilon\left(1 - \frac{\beta_s{}^2}{\beta^2}\right) = j\omega\varepsilon\frac{\beta_z{}^2}{\beta^2}\mathbf{E}_s\end{aligned} \tag{3.15}$$

Letting $\beta_z = \beta\cos\theta$, then the above can be written as

$$\nabla_z \times \mathbf{H}_s = j\omega\varepsilon\cos^2\theta\mathbf{E}_s \tag{3.16}$$

Now looking at (3.4) again, assuming $E_z = 0$, equating transverse components, we have

$$\nabla_z \times \mathbf{E}_s = -j\omega\mu\mathbf{H}_s \tag{3.17}$$

More explicitly, we can rewrite (3.16) and (3.17) the above as

$$\frac{\partial}{\partial z}\hat{z} \times \mathbf{H}_s = j\omega\varepsilon\cos^2\theta\mathbf{E}_s \tag{3.18}$$

$$\frac{\partial}{\partial z}\hat{z} \times \mathbf{E}_s = -j\omega\mu\mathbf{H}_s \tag{3.19}$$

The above resembles the telegrapher's equation. We can multiply (3.19) by $\hat{z}\times$ to get

$$\frac{\partial}{\partial z}\mathbf{E}_s = j\omega\mu\hat{z} \times \mathbf{H}_s \tag{3.20}$$

Now (3.18) and (3.20) look even more like the telegrapher's equation. We can have $\mathbf{E}_s \to$, $\hat{z} \times \mathbf{H}_s \to -I$. $\mu \to L$, $\varepsilon\cos^2\theta \to C$, and the above resembles the telegrapher's equations, or that the waveguide problem is homomorphic to the transmission line problem. The characteristic impedance of this line is then

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\varepsilon\cos^2\theta}} = \sqrt{\frac{\mu}{\varepsilon}}\frac{1}{\cos\theta} = \frac{\omega\mu}{\beta_z} \tag{3.21}$$

Therefore, the TE modes of a waveguide can be mapped into a transmission problem. This can be done, for instance, for the TE_{mn} mode of a rectangular waveguide. Then, in the above

$$\beta_z = \sqrt{\beta^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \tag{3.22}$$

Therefore, each TE_{mn} mode will be represented by a different characteristic impedance Z_0, since β_z are different for different TE_{mn} modes.

3.2 TM Case

This case can be derived using duality principle. Invoking duality, and after some algebra, then the equivalence of (3.18) and (3.20) become

$$\frac{\partial}{\partial z}\mathbf{E}_s = j\omega\mu\cos^2\theta\hat{z}\times\mathbf{H}_s \tag{3.23}$$

$$\frac{\partial}{\partial z}\hat{z}\times\mathbf{H}_s = j\omega\varepsilon\mathbf{E}_s \tag{3.24}$$

To keep the dimension commensurate, we can let $\mathbf{E}_s \to V$, $\hat{z}\times\mathbf{H}_s \to -I$, $\mu\cos^2\theta \to L$, $\varepsilon \to C$, then the above resembles the telegrapher's equations. We can thus let

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu\cos^2\theta}{\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}\cos\theta = \frac{\beta_z}{\omega\varepsilon} \tag{3.25}$$

Please note that (3.21) and (3.25) are very similar to that for the plane wave case, which are the wave impedance for the TE and TM modes, respectively.

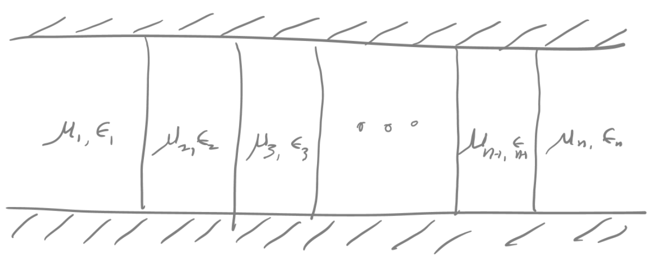

Figure 3:

The above implies that if we have a waveguide of arbitrary cross section filled with layered media, the problem can be mapped to a multi-section transmission line problem, and solved with transmission line methods. When V and I are continuous at a transmission line junction, $\mathbf{E}_s$ and $\mathbf{H}_s$ will also be continuous. Hence, the transmission line solution would also imply continuous $\mathbf{E}$ and $\mathbf{H}$ field solutions.

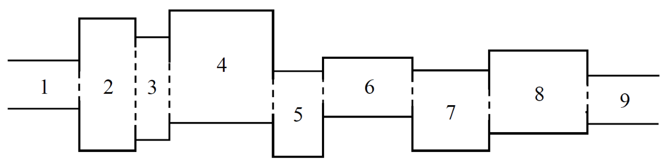

Figure 4:

However, for a multi-junction waveguide show in Figure 4, tangential $\mathbf{E}$ and $\mathbf{H}$ continuous condition cannot be satisfied by a single mode in each waveguide alone: V and I continuous at a transmission line junction will not guarantee the continuity of tangential $\mathbf{E}$ and tangential $\mathbf{H}$ fields at the waveguide junction. Multi-modes have to be assumed in each section in order to match boundary conditions at the junction. Moreover, mode matching method for multiple modes has to be used at each junction. Typically, a single mode incident at a junction will give rise to multiple modes reflected and multiple modes transmitted. The multiple modes give rise to the phenomenon of mode conversion at a junction. Hence, the waveguide may need to be modeled with multiple transmission line.

However, the operating frequency can be chosen so that only one mode is propagating at each section of the waveguide, and the other modes are cutoff or evanescent. In this case, the multiple modes at a junction give rise to localized energy storage at a junction. These energies can be either inductive or capacitive. The junction effect may be modeled by a simple circuit model as shown in Figure 5.

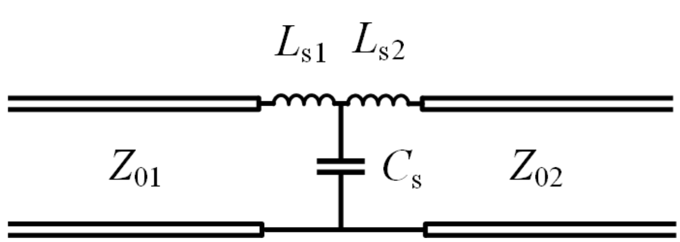

Figure 5: